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LETTER TO THE EDITOR

Quantum global vortex strings in a background field

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Abstract
We consider quantum global vortex string correlation functions, within the Kalb–Ramond framework, in the presence of a background field-strength tensor and investigate the conditions under which this yields a nontrivial contribution to those correlation functions. We show that a background field must be supplemented to the Kalb–Ramond theory in order to correctly describe the quantum properties of the vortex strings. The explicit form of this background field and the associated quantum vortex string correlation function are derived. The complete expression for the quantum vortex creation operator is explicitly obtained. We discuss the potential applicability of our results in the physics of superfluids and rotating Bose–Einstein condensates.

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Global vortex strings appear as nontrivial excitations in many important physical systems described by the relativistic global U(1) Abelian Higgs model in the spontaneously broken phase. Typical examples are the cosmic strings left as remnants of topological phase transitions occurred in the early universe [1]. Superfluid vortices are similar excitations occurring in nonrelativistic systems such as superfluid helium II [2]. The second case is evidently a completely quantum system and, therefore, a full quantum description of the superfluid vortex excitations is necessary. Even for cosmic strings, however, a quantum description is likely to be needed in the early stages of the universe, where quantum effects should be important. Quite interesting works have been devoted to the quantum description of vortices in superfluids [3]. The particular case of vortices in superfluid films has been considered in [4], where a topological coupling between the field associated with the superfluid film and the vortices has been introduced. Yet, a formulation made explicitly in terms of a fully quantized vortex creation operator and its correlation functions would be highly interesting.

More recently, it has been found that superfluid vortices are abundantly generated in rotating Bose–Einstein condensates. These are also deeply quantum systems, and consequently a quantized framework is required for studying the dynamics of such vortices. An intense research activity has been devoted to these systems in the recent years [5].

The relation between global strings and superfluid vortices has been clarified quite a few years ago. It has been shown that classical superfluid vortices could be described as global strings in the presence of a particular nonrelativistic background field [7, 8]. Assuming that this relationship could be extended up to the quantum level, the potential applicability of a quantum theory of global strings to superfluid vortices becomes clear.

In a previous paper, we introduced a creation operator of fully quantized global vortex string states and evaluated its two-point correlation function in the Kalb–Ramond framework [9]. As we argue below, however, this correlation function is not satisfactory. The reason is that its short-distance behaviour does not correspond to what one should expect from a genuine operator creating local states. The fact that these are not normalizable implies that the corresponding correlator should diverge at short distances.

In this work, we evaluate the global vortex string creation operator correlation function in the presence of a background field and investigate the necessary conditions for a nontrivial effect thereof. We show that a background field containing the information that the superfluid state is destroyed along the vortex string should be included, in order for the quantum correlation function to have the nontrivial short-distance behaviour suitable to a genuine operator creating local quantum states. The vortex string correlation functions in the presence of a background are equivalent to the correlation functions of a new vortex string operator, whose form is explicitly obtained.

Global vortex strings are solutions of theories with spontaneously broken global U(1) symmetries. One of the simplest examples is the U(1) Higgs model,

$$\mathcal{L}[\phi] = |\partial_\mu \phi|^2 + m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4, \tag{1}$$

where $\phi = \frac{\rho}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\theta}$ is a complex scalar field. Another example would be the Gross–Pitaevskii equation, which is nonrelativistic [6].

An interesting particular regime is the one when the coupling λ is large and we may approximate ρ by its constant vacuum value $\rho_0 = 2m^2/\lambda$. The only remaining dynamical degree of freedom, in this case, is the scalar (Goldstone boson) field θ:

$$\mathcal{L} = \frac{\rho_0^2}{2}\partial_\mu\theta\partial^\mu\theta. \tag{2}$$

In this framework, global strings are solutions with a nonzero vorticity. This is defined in terms of the vorticity current

$$J^{\mu\nu}(x) \equiv \epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\sigma\theta(x) = 2\pi n \int_{S(C)} \mathrm{d}^2\xi^{\mu\nu}\,\delta^4(x-\xi), \tag{3}$$

where the vortex string coincides with the curve C and $S(C)$ is its world surface. In the above equation, for the vortex current to be nonvanishing, of course, θ must be multivalued. 2π is the vorticity quantum in the units we are working and n is an integer corresponding to the number of vortex quanta. Assuming that the phase θ is defined relatively to the vortex centre, the second part of (3) follows.

An extremely useful equivalent description of this system was obtained [11], in terms of the antisymmetric tensor gauge field $B_{\mu\nu}$, or Kalb–Ramond field, whose field-strength tensor is given by $H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\beta B_{\gamma\alpha} + \partial_\gamma B_{\alpha\beta}$ [12]. The connection is made by writing the U(1) Higgs current as the topological current of the Kalb–Ramond field, namely,

$$J_\mu = \tfrac{1}{6}\epsilon_{\mu\nu\alpha\beta}H^{\nu\alpha\beta} = \rho^2\partial_\mu\theta. \tag{4}$$

It has been shown that in the regime where the field ρ has a constant value ρ_0, the Higgs Lagrangian (1), in the presence of a string configuration associated with the current (3), can be described by [11]

$$\mathcal{L}[B_{\mu\nu}] = \frac{1}{12\rho_0^2} H^2_{\mu\nu\alpha} + \frac{1}{2} B_{\mu\nu} J^{\mu\nu}. \tag{5}$$

The vorticity flux along a surface R is then given by (from now on, we assume that the i, j, k components run over spatial indices)

$$\Phi_R = \int_R \mathrm{d}^2 x^i \, J^{i0}(\vec{x}, t) = \int_R \mathrm{d}^2 x^i \, \partial_j \Pi^{ji}, \tag{6}$$

where Π^{ij} is the momentum canonically conjugate to the Kalb–Ramond field.

An important application of this theory is in the description of superfluid systems. In these, the superfluid density is given by ρ^2 and the superfluid velocity by $\vec{\nabla}\theta$. The superfluid current, therefore, is $\vec{j} = \rho^2 \vec{\nabla}\theta$. This is the spatial component of $j_\mu = \rho^2 \partial_\mu \theta = -\mathrm{i}\phi^* \overleftrightarrow{\partial}_\mu \phi$, namely, the U(1) current of the Higgs model (1). The following point, however, is crucial for using the Kalb–Ramond field in the description of the superfluid. It has been demonstrated that in order to correctly describe the classical physical properties of vortices in superfluid systems, one should add to the field-strength tensor $H^{\mu\nu\alpha}$ a constant nonrelativistic background field [7, 8]

$$\bar{H}^{\mu\nu\alpha} = \begin{cases} \sqrt{\rho_0}\epsilon^{ijk}, & \mu, \nu, \alpha = i, j, k \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The reason why global strings differ from superfluid vortices is that they live in a Lorentz invariant vacuum and, consequently, there is no circulation of energy–momentum around them. Superfluid vortices, conversely, do have a fluid flux circulating them. It has been shown, however, that in the presence of the nonrelativistic background (7), the global string spins around its axis and superfluid vortices are equivalent to spinning global strings [7, 8]. This description is an alternative to the Gross–Pitaevskii formulation that would lead to the nonrelativistic version of (2) in the (constant ρ) incompressible regime (see (25)).

In a previous paper [9], using a general method of quantization of topological excitations [10], we have introduced a quantum global string creation operator, $\sigma(C, t)$, which acting on the vacuum creates a quantum string state at the curve C on the instant t. It is given by

$$\sigma(C, t) = \exp\left\{ \frac{\mathrm{i}a}{2} \int_{T(C)} \mathrm{d}^2 \xi_{\mu\nu} \frac{\partial_\alpha H^{\alpha\mu\nu}}{-\Box} \right\} = \exp\left\{ \frac{-\mathrm{i}a}{2} \int_{T(C)} \mathrm{d}^2 \xi_{ij} \, B^{ij} + \text{gauge terms} \right\}. \tag{8}$$

In the above expressions, a is an arbitrary real number and $T(C)$ is a spacelike surface bounded by the closed string at C. $\mathrm{d}^2\xi_{ij}$ is the surface element of $T(C)$, the directions i, j being along the surface.

We may write (8) as

$$\sigma(C, t) = \exp\left\{ \frac{1}{2} \int \mathrm{d}^4 x \, H_{\alpha\mu\nu} \tilde{C}^{\alpha\mu\nu} \right\}, \tag{9}$$

where

$$\tilde{C}^{\alpha\mu\nu} = \partial^\alpha \tilde{C}^{\mu\nu}, \qquad \tilde{C}^{\mu\nu} = \mathrm{i}a \int_{T(C)} \mathrm{d}^2 \xi_{\mu\nu} \frac{1}{-\Box}(z - \xi). \tag{10}$$

From (6) and (8), we can show [9] that

$$\Phi_R |\sigma(C, t)\rangle = a |\sigma(C, t)\rangle, \tag{11}$$

provided C pierces R. This demonstrates that indeed the quantum string operator creates eigenstates of the vorticity operator with eigenvalue a, which we may choose equal to $2\pi n$ (note that Φ_R and also a are dimensionless).

The Euclidean two-point correlation function of the quantum string operator can be written as the functional integral [9]:

$$\langle\sigma(C_x)\sigma^\dagger(C_y)\rangle = Z^{-1}\int DB_{\mu\nu}\exp\left\{-\int d^4z\left[\frac{1}{12\rho_0^2}H^{\mu\nu\alpha}H_{\mu\nu\alpha} + \frac{1}{6}\tilde{C}_{\mu\nu\alpha}(z;x,y)H^{\mu\nu\alpha} + \frac{\rho_0^2}{6}\tilde{C}^{\mu\nu\alpha}\tilde{C}_{\mu\nu\alpha}\right]\right\}, \tag{12}$$

where Z is the vacuum functional, $\tilde{C}_{\mu\nu\alpha}(z;x,y) = \tilde{C}_{\mu\nu\alpha}(z;x) - \tilde{C}_{\mu\nu\alpha}(z;y)$, corresponding to strings in C_x and C_y. The last term in the exponent is a renormalization counterterm, introduced in order to guarantee locality of the correlation function, which thereby becomes surface independent. An arbitrary n-point quantum global string correlation function would be obtained by just inserting additional external fields $\tilde{C}_{\mu\nu\alpha}(z;x_i)$, $i = 1,\ldots,n$, in (11).

In [9], we have calculated the above correlator, at equal times, in the case of a large straight string of length L along the z-direction and piercing the xy-plane at the point $\vec{x}$. The result is

$$\langle\sigma(\vec{x},t)\sigma^\dagger(\vec{y},t)\rangle = \exp\left\{-\frac{La^2\rho_0^2}{8\pi}|\vec{x}-\vec{y}|\right\}. \tag{13}$$

From the large-distance behaviour of this expression, we can obtain the quantum string creation energy. This is given by $E(L) = \frac{La^2\rho_0^2}{8\pi}$, which means that the string energy per unit length is a constant parametrized by the vorticity of the quantum string and the superfluid density.

Even though the large-distance behaviour of expression (13) is suitable and reveals the energy of the objects being created by σ, its short distance is unsatisfactory. Observe that

$$\lim_{\vec{x}\to\vec{y}}\langle\sigma(\vec{x},t)\sigma^\dagger(\vec{y},t)\rangle = ||\sigma(\vec{x},t)\rangle|^2 = 1. \tag{14}$$

Remember, however, that genuine local states created by a quantized field are not normalizable. Hence, the two-point correlator of a genuine quantum global string creation operator should diverge at short distances. In what follows, we show that this shortcoming can be eliminated by the introduction of a particular background Kalb–Ramond field-strength tensor in (12). By the way, as we have seen before, the correct classical description of superfluid vortices also requires the inclusion of a background field given by (7).

Following the above reasoning, instead of (12), we are going to evaluate

$$\langle\sigma(C_x)\sigma^\dagger(C_y)\rangle_{\bar{H}} = Z^{-1}\int DB_{\mu\nu}\exp\left\{-\int d^4z\left[\frac{1}{12\rho_0^2}(H^{\mu\nu\alpha}+\bar{H}^{\mu\nu\alpha})(H_{\mu\nu\alpha}+\bar{H}^{\mu\nu\alpha}) + \frac{1}{6}\tilde{C}_{\mu\nu\alpha}(z;x,y)(H^{\mu\nu\alpha}+\bar{H}^{\mu\nu\alpha}) + \frac{\rho_0^2}{6}\tilde{C}^{\mu\nu\alpha}\tilde{C}_{\mu\nu\alpha}\right]\right\}, \tag{15}$$

where the background field strength $\bar{H}^{\mu\nu\alpha}$ is *a priori* non-specified.

The functional integral over the $B_{\mu\nu}$ field is quadratic and can be performed by following the same steps used for evaluating (12) [9]. The result is

$$\langle\sigma(C_x)\sigma^\dagger(C_y)\rangle_{\bar{H}} = \exp\left\{F_{\bar{H}=0}(C_x,C_y) - \frac{\rho_0^2}{2}\int d^4z\, d^4z' \times\left\{\left[\frac{1}{6\rho_0^2}\epsilon^{\alpha\beta\gamma\sigma}\partial_\sigma\bar{H}_{\alpha\beta\gamma}(z)\right]\left[\frac{1}{6\rho_0^2}\epsilon^{\mu\nu\lambda\rho}\partial'_\rho\bar{H}_{\mu\nu\lambda}(z')\right] + [\epsilon^{\alpha\beta\gamma\sigma}\partial_\sigma\tilde{C}^{\alpha\beta\gamma}(z)] \times\left[\frac{1}{6\rho_0^2}\epsilon^{\mu\nu\lambda\rho}\partial'_\rho\bar{H}_{\mu\nu\lambda}(z')\right]\right\}\left[\frac{1}{-\Box}\right](z-z')\right\}. \tag{16}$$

In this expression, $F_{\bar{H}=0}(C_x, C_y)$ is the result of the calculation without any background performed in [9] and

$$\left[\frac{1}{-\Box}\right](x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{k^2} = \frac{1}{4\pi^2|\vec{x}|^2}. \tag{17}$$

The second term in the exponent in (16) vanishes because $\tilde{C}^{\alpha\beta\gamma}$ is a derivative (see equation (10)). Accordingly, for the first term to be different from zero, the background field should *not* be of the form $\bar{H}_{\alpha\beta\gamma} = \partial_\alpha\bar{B}_{\beta\gamma} + \partial_\beta\bar{B}_{\gamma\alpha} + \partial_\gamma\bar{B}_{\alpha\beta}$. In this case, the first term in (16) would also vanish. It is easy to see that when the background has this form it gives no contribution to (15): by shifting the functional integration variable as $B_{\mu\nu} \rightarrow B_{\mu\nu} + \bar{B}_{\mu\nu}$, we would eliminate the background from (15). It is also clear from (16) that a constant background such as (7) would have no effect in the quantum string correlation function.

The condition for the background field to give a nonzero contribution to the correlator (15) is that it should *not* satisfy Bianchi's identity, namely,

$$\tfrac{1}{6}\epsilon^{\alpha\beta\gamma\sigma}\partial_\sigma\bar{H}_{\alpha\beta\gamma} \neq 0. \tag{18}$$

According to (4), however, this means that the superfluid current would no longer be conserved in the presence of the background $\bar{H}_{\alpha\beta\gamma}$. This observation provides us the clue for obtaining a suitable form for the background field $\bar{H}_{\alpha\beta\gamma}$. Along a vortex, the superfluid phase is destroyed and, consequently, the superfluid current is no longer conserved because of depletion. The same thing should happen for a quantum vortex string. Hence, when computing the quantum string correlator (15), we must introduce a background field such that Bianchi's identity is not satisfied along the vortices. The simplest configuration satisfying this requirement is

$$\frac{1}{6}\epsilon^{\alpha\beta\gamma\sigma}\partial_\sigma\bar{H}_{\alpha\beta\gamma} = \rho_0\, a\oint_{C_x-C_y}\mathrm{d}\xi^\mu\,\hat{n}_\mu(\xi)\delta^4(z-\xi), \tag{19}$$

where $\hat{n}_\mu(\xi)$ is a unit vector tangent to the string at the point ξ. The background field strength satisfying (19) is

$$\bar{H}^{\alpha\beta\gamma} = \rho_0\, a\epsilon^{\alpha\beta\gamma\nu}\int_{T(C_x)-T(C_y)}\mathrm{d}^2\xi_{\mu\nu}\,\hat{n}^\mu(\xi)\delta^4(z-\xi), \tag{20}$$

where $T(C)$ is a spacelike surface bounded by C. Inserting (20) in (16) and performing the z and z' integrals, we obtain the following expression for the $\bar{H}$ contribution to the correlation function:

$$\exp\left\{\frac{-a^2}{2}\sum_{i,j=1}^{2}\lambda_i\lambda_j\oint_{C_i}\mathrm{d}\xi^\mu\oint_{C_j}\mathrm{d}\eta^\nu\,\hat{n}_\mu(\xi)\hat{n}_\nu(\eta)\left[\frac{1}{-\Box}\right](|\xi-\eta|)\right\}. \tag{21}$$

In this expression, $\lambda_i = \pm 1$, corresponding to C_x and C_y, respectively. The self-interaction terms *i*–*i* are eliminated by a renormalization of the string operator σ.

Considering again the previous situation of a large straight string along the z-axis and piercing the xy-plane at $\vec{x}$, we may follow the procedure described in [13] to evaluate (21). Including the $\bar{H} = 0$ contribution, calculated before, we get

$$\langle\sigma(\vec{x},t)\sigma^\dagger(\vec{y},t)\rangle_{\bar{H}} = \exp\left\{\frac{La^2}{|\vec{x}-\vec{y}|} - \frac{La^2\rho_0^2}{8\pi}|\vec{x}-\vec{y}|\right\}. \tag{22}$$

We now have

$$\lim_{\vec{x}\to\vec{y}}\langle\sigma(\vec{x},t)\sigma^\dagger(\vec{y},t)\rangle = |\langle\sigma(\vec{x},t)\rangle|^2 \rightarrow \infty, \tag{23}$$

which is the expected behaviour of a genuine creation operator of local states. The large-distance behaviour is the same as before, yielding a string tension $\frac{a^2\rho_0^2}{8\pi}$.

In the same way that the Kalb–Ramond theory should be supplemented by the addition of the background field (7) in order to correctly describe the classical properties of vortex strings [7], it must be supplemented by the introduction of the background field (20) in order to describe the quantum properties of these excitations. It would be interesting to study the behaviour of $\langle\sigma\rangle$, in order to investigate the possible relationship between the two background fields.

An alternative interpretation that is quite appealing is to regard (15) and (22) as the correlator of a new vortex string creation operator $\Sigma \equiv \sigma\mu$. From (15) and (20), dropping the renormalization terms, we find that the operator μ is given by

$$\mu(C,t) = \exp\left\{\frac{\mathrm{i}a}{2\rho_0}\int_{T(C)} \mathrm{d}^2\xi_{\mu\nu}\,\hat{n}^\nu\epsilon^{\mu\alpha\beta\gamma}\partial_\alpha B_{\beta\gamma}\right\} = \exp\left\{\frac{\mathrm{i}a}{2\rho_0}\int_{T(C)} \mathrm{d}^2\xi_{ij}\,\hat{n}^j\epsilon^{\mathrm{i}kl}\Pi^{kl}\right\}, \qquad (24)$$

where Π^{ij} is the momentum canonically conjugate to the Kalb–Ramond field B^{ij}.

Equation (12) would also be satisfied by the states $|\Sigma(C,t)\rangle$, since μ commutes with Φ_R. This shows that Σ indeed is a quantum global string creation operator. Note that it resembles the Mandelstam operator of 2D field theories [14]. This suggests a parallelism between the systems studied here and the two-dimensional ones where the Mandelstam operator is relevant, such as the sine-Gordon theory and the Coulomb gas.

Expression (22) for the vortex string correlation function and the new quantum vortex creation operator obtained thereof should be relevant in the study of the dynamics of quantum vortices, vortex lattice formation and in the vortex nucleation problem in superfluid helium II and in rotating Bose–Einstein condensates.

Let us remark, finally, that the relevant physics of superfluids is described by the Gross–Pitaevskii equation [6], which is of first order in time. In this case, the constant ρ regime of the fluid would be described by the nonrelativistic version of (14), namely,

$$\mathcal{L} = \frac{\rho_0^2}{2}[\mathrm{i}\theta\partial_0\theta + \partial_i\theta\partial_i\theta]. \qquad (25)$$

Accordingly, in order to construct the vortex operator corresponding to σ or Σ directly in the Gross–Pitaevskii framework, we should consider the nonrelativistic version of (5), (12), (14) and (15). We intend to pursue this in a future publication.

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